The inner viscous solution for the core of a leading-edge vortex

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In an earlier paper Hall (1961) proposed a simplified model for the vortex core formed over a slender delta wing at incidence by the rolling-up of the shear layer that separates from a leading edge. This model enabled an outer inviscid solution and an inner viscous solution for the core to be obtained from the equations of motion. However the procedure used for the inner solution led to a number of defects: in particular, the matching of the inner and outer solutions seemed unsatisfactory. In the present paper the defects are avoided by using a different procedure. The first approximation, in the sense of boundary-layer theory, is sought. A solution, in special variables, is obtained which is in the form of an asymptotic expansion containing inverse powers of the logarithm of a Reynolds number. The leading terms of the expansion are computed, and the results confirm that the inner and outer solutions are properly matched.

1. Introduction

The simplified model of the vortex core proposed by Hall (1961) and adopted here is as follows. The core is geometrically slender, the fluid is incompressible, and the velocity and pressure fields are steady and axially symmetric. There are two distinctive properties: (1) the flow is continuous, i.e. it includes no vortex sheet, so that it must be rotational to allow a convection of vorticity; and (2) diffusion of vorticity is confined to a relatively slender inner core, i.e. the kinematic viscosity is small. The latter reduces the problem of solving the Navier–Stokes equations for the core to that of obtaining first an inviscid outer solution for the convective part—for which the inner core is ignored—and then, using this outer solution to specify boundary conditions, obtaining a viscous inner solution for the inner core. In addition to the above properties the velocity and pressure fields are, for the outer solution, taken to be conical.

Given appropriate conditions on the outside edge of the core and zero radial velocity on the axis of symmetry, the outer solution is easily obtained explicitly. The resulting expressions for the velocity components and the pressure are in simple logarithmic form. It can be deduced, by substitution of this solution into the full Navier–Stokes equations, that a diffusive viscous inner core must exist. This outer solution will be taken as a starting-point in the present paper.

For his inner solution Hall (1961) made assumptions of boundary-layer type but chose variables and boundary conditions appropriate for finite Reynolds numbers and, in addition, made a number of approximations based on the boundary conditions. Two of the latter, an Oseen condition of nearly constant axial velocity and the use of the outer solution for the radial velocity, were justified only *a posteriori* as being adequate for practical purposes where the kinematic viscosity ν is small but not zero. The results were in qualitative agreement with experimental results but it was natural to ask whether they could be justified more rigorously. Hall's approach made it difficult to examine the limiting behaviour of the solution as $\nu \to 0$, and he was led to conclude that it was necessary, even in the limit $\nu = 0$, to join the inner and outer solutions at a finite rather than infinite value of the ratio of the distance from the axis to the square root of the kinematic viscosity, and that this implied that the solution remains approximate in the limit $\nu \to 0$, unlike boundary-layer solutions which become exact. It is now clear that these conclusions are incomplete and misleading.

In the present paper these drawbacks are removed by using a rigorously mathematical approach in which the variables and the form of the solution are chosen from the start specifically for the limit $\nu \to 0$. The procedure is as follows. First, the Navier–Stokes equations are simplified by making approximations of boundary-layer type. Next the appropriate independent variables are defined; the outer solution is expressed in terms of these variables and, with this as a guide, an asymptotic expansion for the inner solution is set down. The expansion is then substituted in the equations of motion and, by equating terms of like magnitude, a set of ordinary differential equations is obtained which, in association with appropriate boundary conditions, yields a solution which approaches the outer solution with increasing distance from the axis. The leading equations of this set have been solved numerically at the Royal Aircraft Establishment and a selection of the results is tabulated below.

The limit of vanishing viscosity manifests itself through two non-dimensional numbers ζ, χ defined in equations (9) below, of which ζ depends on the distances along and from the axis of symmetry while χ depends on distance along the axis. The introduction of the number χ brings out an unusual complication in that the boundary conditions on the inner solution that are set by the outer solution themselves depend on χ and thus on ν; moreover, the inner solution is found as a series of descending powers of χ. In conventional problems of boundary-layer or shear-layer type corresponding variables occur, but the boundary conditions set by the outer solution are independent of ν and the range of validity of the inner solution in terms of the variable corresponding to χ is much larger than in the present case, where $\chi \to \infty$ in the limit $\nu \to 0$.

The physical implications of the above are equally unusual. The trend $\chi \to \infty$ in the limit $\nu \to 0$ corresponds to an approach to infinite velocity and negatively infinite pressure along the axis of the vortex core as the viscosity becomes vanishingly small. Thus the present model of the core is unrealistic in the extreme condition $\nu \to 0$, for if viscous effects are sufficiently small they are necessarily accompanied in reality by compressibility effects. Of course for finite ν compressibility effects, which depend on temperature, can be negligible while viscous effects are small.

An earlier, less complete, discussion of the properties of the inner solution has already been given by one of us (Stewartson 1961) and forms the basis of the present paper.

2. The approximation of boundary-layer type: the first limiting process

When the velocity and pressure fields are axially symmetric, the Navier–Stokes equations for the steady flow of an incompressible fluid are, in cylindrical co-ordinates (r, x),

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial r}+\frac{w}{r} = 0, \tag{1a}$$

$$u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\nabla^2 u, \tag{1b}$$

$$u\frac{\partial v}{\partial x}+w\frac{\partial v}{\partial r}+\frac{vw}{r} = \nu\left(\nabla^2 v-\frac{v}{r^2}\right), \tag{1c}$$

$$u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial r}-\frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left(\nabla^2 w-\frac{w}{r^2}\right), \tag{1d}$$

where
$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial x^2},$$

u and w are the axial and radial velocity components, v is the circumferential velocity component, and p, ρ and ν are the pressure, density and kinematic viscosity, respectively.

The velocity components and the pressure in the inviscid outer solution will be denoted by u_i, v_i, w_i and p_i respectively. The boundary conditions for the outer solution are taken to be

$$r = 0,\ w_i = 0; \quad r = ax,\ u_i = U,\ v_i = V,\ p_i = P, \tag{2}$$

where $r = ax$ is the conical, outside edge of the core, so that a, U, V and P are all constants. With these boundary conditions, equations (1) give, for an inviscid flow in which the velocity and pressure fields are conical,

$$\left.\begin{aligned} u_i &= U - U\alpha\log(r/ax),\\ v_i &= [V^2 - U^2\alpha^2\log(r/ax)]^{\frac{1}{2}},\\ w_i &= -\tfrac{1}{2}U\alpha r/x,\\ p_i - P &= \rho V^2\log(r/ax) - \tfrac{1}{2}\rho U^2\alpha^2\log^2(r/ax), \end{aligned}\right\} \tag{3}$$

where
$$\alpha = (1+2V^2/U^2)^{\frac{1}{2}} - 1 > 0,$$

and the slenderness condition $(r/x)^2 \gg 1$ has been assumed. The effect on the equations (1) of this condition is simply that the terms $u\,\partial w/\partial x$ and $w\,\partial w/\partial r$ may be neglected in comparison with the remainder of the terms in (1*d*). We shall refer to equations (3) as the outer solution. It must break down as $r \to 0$ because the viscous terms in equations (1) then become important. In fact, substitution of the outer solution into (1) shows that the viscous terms become important when

$$r = O[x(U\alpha x/\nu)^{-\frac{1}{2}}]. \tag{4}$$

For small distances from the axis the outer solution (3) gives

$$\left.\begin{aligned} u_i &= O[U\alpha \log(r/x)], & \partial u_i/\partial r &= O(U\alpha/r),\\ v_i &= O[U\alpha \log^{\frac{1}{2}}(r/x)], & w_i &= O(U\alpha r/x).\end{aligned}\right\} \tag{5}$$

Now the proper boundary conditions for the viscous inner core are

$$\left.\begin{aligned} &r = 0, \quad \partial u/\partial r = v = w = 0,\\ \text{and, from (4),}\quad &(r/x)(U\alpha x/\nu)^{\frac{1}{2}} \to \infty, \quad u \to u_i, \quad v \to v_i, \quad p \to p_i.\end{aligned}\right\} \tag{6}$$

From the relations (5) and the boundary conditions (6) orders of magnitude can be assigned to the terms in the Navier–Stokes equations (1) as in boundary-layer theory. On equating the magnitudes of viscous and inertia terms it is found that $\nu = O(U\alpha r^2/x)$, as might be expected from (4). The differences in magnitude of the terms in (1) are found to depend on two factors, $U\alpha x/\nu$ and $\log(r/x)$. Only the first is considered at this stage. A boundary-layer approximation to equations (1) is made by taking the limit

$$U\alpha x/\nu \to \infty \quad \text{with} \quad (r/x)(U\alpha x/\nu)^{\frac{1}{2}} \quad \text{fixed}. \tag{7}$$

This means that advantage is taken of the fact that when ν is sufficiently small the viscous region will be very slender and changes in the radial direction will be much more marked than in the axial direction. The resulting equations are

$$\left.\begin{aligned} &\frac{\partial u}{\partial x}+\frac{\partial w}{\partial r}+\frac{w}{r} = 0,\\ &u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right),\\ &u\frac{\partial v}{\partial x}+w\frac{\partial v}{\partial r}+\frac{wv}{r} = \nu\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}\right),\\ &\frac{v^2}{r} = \frac{1}{\rho}\frac{\partial p}{\partial r}.\end{aligned}\right\} \tag{8}$$

As usual in boundary-layer theory the equations are still a highly non-linear set of partial differential equations: the boundary-layer approximation has reduced the order of the equations, by removing the second derivatives with respect to x, and has virtually uncoupled one equation from the others. Observe that equations (8) include the equations governing the outer solution (3) as the limiting case $\nu = 0$, and that the limit (7) happens to be equivalent to the slenderness condition $(r/x)^2 \to 0$ of the outer solution.

3. Derivation of the ordinary differential equations: the second limiting process

The procedure in this section will be to define new independent variables, to substitute these into the outer solution, to use the resulting expressions to set down an asymptotic expansion for the inner solution and, finally, to substitute this asymptotic expansion into equations (8) and collect terms of like magnitude to obtain the required ordinary differential equations.

After trial it is found that appropriate independent variables for the inner viscous flow are

$$\zeta = \frac{r}{x}\left(\frac{U\alpha\chi x}{\nu}\right)^{\frac{1}{2}}, \quad \chi = \log\left[\alpha\left(\frac{U\alpha x}{\nu}\right)^{\frac{1}{2}}\right]. \tag{9}$$

Note that the definition of ζ is different from that of Hall's earlier work in not requiring that any boundary be assigned to the viscous region: ζ is chosen specifically with the limit of vanishing viscosity in mind; the 'edge' of the viscous region is given by $\zeta/\chi^{\frac{1}{2}} \to \infty$, and the edge of the core as a whole by $\zeta = \chi^{\frac{1}{2}} e^{\chi}$. Note also that the region where viscous effects are important is of thickness $\zeta = O(\chi^{\frac{1}{2}})$, from equation (4), and is not of unit order of magnitude, as is usual with the corresponding boundary-layer variable. The second limiting process of the inner solution, to be applied later in this section, is

$$\chi \to \infty \quad \text{with } x \text{ fixed}. \tag{10}$$

Since $\chi \to \infty$ only very slowly as $\nu \to 0$ it might be expected that for the practical application to cases in which $\nu \neq 0$ several terms of an expansion in inverse powers of χ would be required.

In terms of ζ and χ the outer solution (3) can be rewritten

$$\left.\begin{aligned} u_i &= U\alpha\chi\left[1+\frac{\log\chi}{2\chi}+\frac{1}{\chi}(-\log\zeta+1/\alpha)\right],\\ v_i &= U\alpha\chi^{\frac{1}{2}}\left[1+\frac{\log\chi}{4\chi}+\frac{1}{\chi}(-\tfrac{1}{2}\log\zeta+\tfrac{1}{4}+1/2\alpha)+\dots\right],\\ w_i &= -\frac{U\alpha}{2}\left(\frac{U\alpha\chi x}{\nu}\right)^{-\frac{1}{2}}\zeta,\\ p_i-P &= -\tfrac{1}{2}\rho U^2\alpha^2\chi^2\left[1+\frac{\log\chi}{\chi}+\frac{1}{\chi}(-2\log\zeta+1+2/\alpha)+\frac{\log^2\chi}{4\chi^2}\right.\\ &\qquad\left.+\frac{\log\chi}{\chi^2}(-\log\zeta+\tfrac{1}{2}+1/\alpha)-\frac{1}{\chi^2}\{\log^2\zeta-(1+2/\alpha)\log\zeta\}\right].\end{aligned}\right\} \tag{11}$$

Note that the expansion for v_i holds provided

$$|\tfrac{1}{2}\log\chi-\log\zeta+\tfrac{1}{2}+1/\alpha| < \chi,$$

that is, provided

$$e^{\frac{1}{2}+1/\alpha}\chi^{\frac{1}{2}}e^{-\chi} < \zeta < e^{\frac{1}{2}+1/\alpha}\chi^{\frac{1}{2}}e^{\chi}.$$

These suggest putting, for the inner solution,

$$\left.\begin{aligned} u &= U\alpha\chi F(\zeta,\chi), \quad v = U\alpha\chi^{\frac{1}{2}}G(\zeta,\chi),\\ w &= -\tfrac{1}{2}U\alpha(U\alpha\chi x/\nu)^{-\frac{1}{2}}\zeta H(\zeta,\chi), \quad p-P = -\tfrac{1}{2}\rho U^2\alpha^2\chi^2 C(\zeta,\chi),\end{aligned}\right\} \tag{12}$$

where
$$
\left.\begin{aligned}
F &= 1+\frac{\log\chi}{2\chi}+\frac{1}{\chi}F_1(\zeta)+\frac{\log\chi}{\chi^2}F_2(\zeta)+\frac{1}{\chi^2}F_3(\zeta)+\dots,\\
G &= G_0(\zeta)+\frac{\log\chi}{\chi}G_1(\zeta)+\frac{1}{\chi}G_2(\zeta)+\dots,\\
H &= \tfrac{1}{2}+H_0(\zeta)+\frac{\log\chi}{\chi}H_1(\zeta)+\frac{1}{4\chi}+\frac{1}{\chi}H_2(\zeta)+\dots,\\
C &= 1+\frac{\log\chi}{\chi}+\frac{1}{\chi}C_1(\zeta)+\frac{\log^2\chi}{4\chi^2}+\frac{\log\chi}{\chi^2}C_2(\zeta)+\frac{1}{\chi^2}C_3(\zeta)+\dots,
\end{aligned}\right\}\tag{13}
$$

and the boundary conditions are, with derivatives with respect to ζ, denoted by an accent,

$$
\left.\begin{aligned}
\zeta = 0,\quad & F_1' = F_2' = F_3' = \dots = 0,\\
& G_0 = G_1 = G_2 = \dots = 0,\\
& \zeta H_0 = \zeta H_1 = \zeta H_2 = \dots = 0,\\
\zeta/\chi^{\frac{1}{2}}\to\infty,\quad & F_1\to -\log\zeta+1/\alpha,\ F_2,\ F_3,\ \dots,\to 0,\\
& G_0\to 1,\quad G_1\to\tfrac{1}{4},\quad G_2\to -\tfrac{1}{2}\log\zeta+\tfrac{1}{4}+1/2\alpha,\quad \dots,\\
& C_1\to -2\log\zeta+1+2/\alpha,\quad C_2\to -\log\zeta+\tfrac{1}{2}+1/\alpha,\\
& C_3\to \log^2\zeta-(1+2/\alpha)\log\zeta,\quad \dots .
\end{aligned}\right\}\tag{14}
$$

If the expansion (13) is compatible with the equations of motion (8) and the solutions for the 'F's, 'G's, 'H's and 'C's satisfy the boundary conditions (14), these solutions will constitute a valid inner solution for the vortex core.

Substitution of the expressions (12) in equations (8) gives

$$
\left.\begin{aligned}
&\frac{F}{\chi}+\frac{\partial F}{\partial\chi}-\left(1-\frac{1}{2\chi}\right)\zeta\frac{\partial F}{\partial\zeta}-\frac{2H}{\chi}-\frac{\zeta}{\chi}\frac{\partial H}{\partial\zeta}=0,\\
&\frac{F^2}{2\chi}+\frac{F}{2}\frac{\partial F}{\partial\chi}-\left(1-\frac{1}{2\chi}\right)\frac{\zeta F}{2}\frac{\partial F}{\partial\zeta}-\frac{\zeta H}{2\chi}\frac{\partial F}{\partial\zeta}=\frac{C}{2\chi}+\frac{1}{4}\frac{\partial C}{\partial\chi}\\
&\qquad\qquad -\left(1-\frac{1}{2\chi}\right)\frac{\zeta}{4}\frac{\partial C}{\partial\zeta}+\frac{\partial^2F}{\partial\zeta^2}+\frac{1}{\zeta}\frac{\partial F}{\partial\zeta},\\
&\frac{FG}{4\chi}+\frac{F}{2}\frac{\partial G}{\partial\chi}-\left(1-\frac{1}{2\chi}\right)\frac{\zeta F}{2}\frac{\partial G}{\partial\zeta}-\frac{HG}{2\chi}-\frac{\zeta H}{2\chi}\frac{\partial G}{\partial\zeta}=\frac{\partial^2G}{\partial\zeta^2}+\frac{1}{\zeta}\frac{\partial G}{\partial\zeta}-\frac{G}{\zeta^2},\\
&\chi\frac{\partial C}{\partial\zeta}=-\frac{2G^2}{\zeta}.
\end{aligned}\right\}\tag{15}
$$

These equations are exactly equivalent to (8): the second limiting process (10) is only now to be applied. It can be seen from the expansion for F in equations (13) that the process of taking the limit $\chi\to\infty$ may be interpreted as an exploitation of the idea that for sufficiently small values of the kinematic viscosity the axial velocity should be nearly constant across the viscous region.

Substitution of the expansions (13) in equations (15) yields a set of lengthy equations in F_1, F_2, F_3, …, G_0, G_1, …, etc. If the 'F's, 'G's, 'H's and 'C's, their derivatives, and ζ are all supposed to be of unit order of magnitude, and if the second limiting process (10) is applied, the terms of the equations will appear in descending orders of magnitude. On equating, in turn, the terms of

$$O(1), \quad O[(1/\chi)\log\chi], \quad O(1/\chi)$$

and so on, the following set of ordinary differential equations is obtained.

$$\left.\begin{aligned}
&\zeta H_0'+2H_0 = -\zeta F_1', \quad \zeta H_1'+2H_1 = -\zeta F_2', \\
&\zeta H_2'+2H_2 = \tfrac{1}{2}\zeta F_1'-\zeta F_3', \quad \ldots, \\
&F_1''+(\tfrac{1}{2}\zeta+1/\zeta)F_1' = \tfrac{1}{4}\zeta C_1', \quad F_2''+(\tfrac{1}{2}\zeta+1/\zeta)F_2' = \tfrac{1}{4}\zeta C_2'-\tfrac{1}{4}\zeta F_1', \\
&F_3''+(\tfrac{1}{2}\zeta+1/\zeta)F_3' = \tfrac{1}{4}\zeta C_3'-\tfrac{1}{4}C_1-\tfrac{1}{8}\zeta C_1'+\tfrac{1}{2}F_1-\tfrac{1}{2}\zeta F_1F_1'-\tfrac{1}{2}\zeta H_0F_1', \quad \ldots, \\
&G_0''+(\tfrac{1}{2}\zeta+1/\zeta)G_0'-G_0/\zeta^2 = 0, \quad G_1''+(\tfrac{1}{2}\zeta+1/\zeta)G_1'-G_1/\zeta^2 = -\tfrac{1}{4}\zeta G_0', \\
&G_2''+(\tfrac{1}{2}\zeta+1/\zeta)G_2'-G_2/\zeta^2 = -\tfrac{1}{2}\zeta F_1G_0'-\tfrac{1}{2}H_0G_0-\tfrac{1}{2}\zeta H_0G_0', \quad \ldots, \\
&C_1' = -2G_0^2/\zeta, \quad C_2' = -4G_0G_1/\zeta, \quad C_3' = -4G_0G_2/\zeta, \quad \ldots
\end{aligned}\right\} \quad (16)$$

The self-consistency of equations (16) shows that the expansion (13) is, as required, compatible with the equations of motion.

Note that the differential equations (16) are derived on the assumption that $\zeta = O(1)$, and the domain $\zeta = O(1)$ is but a small part, $1/\chi^{\frac{1}{2}}$, of the viscous core. On the other hand, equations (16) yield the leading terms in an expansion, in descending powers of χ, of the solution of equations (8) which include the equations for the outer solution as the limiting case $\nu = 0$, and a solution of (8) is capable of satisfying all the boundary conditions imposed on the outer solution as well as conditons on the axis proper for a viscous flow. It may be expected, therefore, that the solution of equations (16) can be extended beyond $\zeta = O(1)$. This point will be discussed in more concrete terms after the solution has been obtained.

4. The numerical solution

With the exception of G_0 and H_0, all the 'F's, 'G's, 'H's and 'C's depend on the parameter α as well as ζ. It is therefore convenient to introduce subsidiary functions which are universal functions of ζ by putting

$$\left.\begin{aligned}
&F_1 = F_{10}(\alpha)-F_{11}(\zeta), \quad F_2 = F_{20}(\alpha)-2G_{10}(\alpha)F_{11}(\zeta)+F_{21}(\zeta), \\
&F_3 = F_{30}(\alpha)-[2G_{20}(\alpha)-\tfrac{1}{2}]F_{11}(\zeta)+[2F_{10}(\alpha)-1]F_{21}(\zeta) \\
&\qquad -[\tfrac{1}{2}C_{10}(\alpha)-F_{10}(\alpha)]F_{31}(\zeta)+F_{32}(\zeta), \quad \ldots, \\
&G_0 = G_0(\zeta), \quad G_1 = G_{10}(\alpha)G_0(\zeta)-G_{11}(\zeta), \\
&G_2 = G_{20}(\alpha)G_0(\zeta)-[2F_{10}(\alpha)-1]G_{11}(\zeta)-G_{21}(\zeta), \quad \ldots, \\
&H_0 = H_0(\zeta), \quad H_1 = 2G_{10}(\alpha)H_0(\zeta)-H_{11}(\zeta), \\
&H_2 = [2G_{20}(\alpha)-1]H_0(\zeta)-[2F_{10}(\alpha)-1]H_{11}(\zeta) \\
&\qquad +[\tfrac{1}{2}C_{10}(\alpha)-F_{10}(\alpha)]H_{21}(\zeta)-H_{22}(\zeta), \quad \ldots, \\
&C_1 = C_{10}(\alpha)-C_{11}(\zeta), \quad C_2 = C_{20}(\alpha)-2G_{10}(\alpha)C_{11}(\zeta)+C_{21}(\zeta), \\
&C_3 = C_{30}(\alpha)-2G_{20}(\alpha)C_{11}(\zeta)+[2F_{10}(\alpha)-1]C_{21}(\zeta)+C_{31}(\zeta), \quad \ldots
\end{aligned}\right\} \quad (17)$$

The fourteen universal functions defined specifically here have been determined by substituting the expressions (17) into the equations (16) and solving the resulting equations simultaneously by the step-by-step method of Runge and Kutta with a high-speed digital computer. Since the Runge–Kutta method fails at $\zeta = 0$, because of terms in the equations containing the factor $1/\zeta$, the computation was begun at a small value of ζ ($\zeta = 0{\cdot}02$). The initial conditions were determined by formal integration of the equations and application of the boundary conditions at $\zeta = 0$ given in (14). This is straightforward because the solution for G_0 which satisfies all the requisite boundary conditions can be expressed analytically by

$$\begin{aligned} G_0 &= \tfrac{1}{4}\pi^{\frac{1}{2}}\,\zeta e^{-\frac{1}{4}\zeta^2}\,{}_1F_1(\tfrac{3}{2}, 2, \tfrac{1}{4}\zeta^2) \\ &= \tfrac{1}{4}\pi^{\frac{1}{2}}\,\zeta e^{-\frac{1}{4}\zeta^2}\left[1+\frac{\frac{3}{2}}{2}\cdot\tfrac{1}{4}\zeta^2+\frac{\frac{3}{2}\cdot\frac{5}{2}}{2!\,.\,2\,.\,3}(\tfrac{1}{4}\zeta^2)^2+\ldots\right], \end{aligned} \tag{18}$$

where ${}_1F_1$ is a confluent hypergeometric function, so that, for small ζ, G_0 is given by the rapidly converging series

$$G_0 = \tfrac{1}{4}\pi^{\frac{1}{2}}\,(\zeta-\tfrac{1}{16}\zeta^3+\ldots);$$

once G_0 is known the other universal functions can be determined successively. The functions F_{11}, G_0, G_{11}, G_{21}, H_0 and C_{11} are tabulated here for $0 \leqslant \zeta \leqslant 10$ (table 1). These should be adequate for qualitative comparisons with experimental results and for some stability calculations. Even if laminar vortex cores could be found at large Reynolds numbers, no useful quantitative comparisons could be made without a great deal more computation, because the successive terms in the expansion (13) for F, G, H and C decrease in magnitude only very slowly when $\nu \neq 0$. The computed results do, however, confirm the validity of the solution, and this is their chief importance.

It is clear from the expressions (17) that the computation of the universal functions of ζ does not complete the solution. The parameters F_{10}, F_{20}, F_{30}, G_{10}, G_{20}, C_{10}, C_{20} and C_{30} are still to be determined by application of the boundary conditions (14) for $\zeta/\chi^{\frac{1}{2}} \to \infty$. The question of whether the solution is valid for $\zeta/\chi^{\frac{1}{2}} \to \infty$ is set aside till the next section. To determine $F_{10}(\alpha)$, for example, one would write, from the boundary conditions (14) for F_1 and from (17),

$$F_{10}(\alpha) = \underset{\zeta/\chi^{\frac{1}{2}}\to\infty}{\mathrm{Lt}}(F_{11}-\log\zeta)+1/\alpha, \tag{19}$$

and one would then compute F_{11} for larger and larger ζ until further increases in ζ produce no significant change in the difference $F_{11}-\log\zeta$. If $F_{11}-\log\zeta \to D$ then $F_{10} = D+1/\alpha$. This is the procedure usually followed in boundary-layer calculations, and there it is sufficient to take the independent variable (which corresponds to $\zeta/\chi^{\frac{1}{2}}$) no further than say 4 or 5. It is found here that up to and beyond $\zeta = 10$ the difference $F_{11}-\log\zeta$ changes appreciably with increasing ζ. However, the need to extend the computation is avoided by again making use of the analytic solution (18) for G_0. From the asymptotic form of the confluent hypergeometric function the asymptotic solution for G_0 is

$$G_0 = 1-\frac{1}{\zeta^2}-\frac{3}{2\zeta^4}-\frac{15}{2\zeta^6}-\ldots, \quad \text{where} \quad \tfrac{1}{4}\zeta^2 \gg 1.$$

ζ	F_{11}	G_0	G_{11}	G_{21}	H_0	C_{11}
0·0	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000
0·2	0·0000	0·0884	0·0001	0·0001	0·0000	0·0078
0·4	0·0002	0·1755	0·0009	0·0009	0·0001	0·0311
0·6	0·0008	0·2600	0·0029	0·0029	0·0005	0·0691
0·8	0·0023	0·3409	0·0065	0·0065	0·0015	0·1208
1·0	0·0055	0·4171	0·0122	0·0123	0·0036	0·1848
1·2	0·0109	0·4879	0·0201	0·0201	0·0071	0·2594
1·4	0·0191	0·5528	0·0300	0·0300	0·0124	0·3429
1·6	0·0306	0·6115	0·0417	0·0419	0·0197	0·4334
1·8	0·0458	0·6640	0·0550	0·0554	0·0292	0·5293
2·0	0·0649	0·7103	0·0693	0·0701	0·0409	0·6289
2·2	0·0877	0·7507	0·0843	0·0857	0·0547	0·7306
2·4	0·1141	0·7856	0·0993	0·1017	0·0703	0·8334
2·6	0·1483	0·8156	0·1142	0·1179	0·0873	0·9361
2·8	0·1762	0·8410	0·1283	0·1339	0·1055	1·0378
3·0	0·2110	0·8626	0·1416	0·1496	0·1245	1·1380
3·2	0·2475	0·8807	0·1538	0·1649	0·1437	1·2361
3·4	0·2853	0·8959	0·1648	0·1797	0·1629	1·3318
3·6	0·3239	0·9087	0·1746	0·1941	0·1818	1·4249
3·8	0·3629	0·9194	0·1833	0·2080	0·2002	1·5152
4·0	0·4019	0·9284	0·1908	0·2214	0·2177	1·6028
4·2	0·4406	0·9360	0·1973	0·2346	0·2344	1·6876
4·4	0·4789	0·9424	0·2029	0·2474	0·2501	1·7697
4·6	0·5165	0·9479	0·2077	0·2600	0·2648	1·8491
4·8	0·5533	0·9527	0·2118	0·2724	0·2785	1·9260
5·0	0·5893	0·9568	0·2154	0·2847	0·2912	2·0004
5·2	0·6244	0·9603	0·2184	0·2968	0·3030	2·0725
5·4	0·6586	0·9635	0·2211	0·3087	0·3139	2·1423
5·6	0·6919	0·9662	0·2234	0·3205	0·3240	2·2101
5·8	0·7243	0·9687	0·2255	0·3322	0·3333	2·2757
6·0	0·7558	0·9708	0·2273	0·3437	0·3420	2·3395
6·2	0·7865	0·9728	0·2289	0·3551	0·3500	2·4014
6·4	0·8163	0·9745	0·2303	0·3664	0·3574	2·4616
6·6	0·8454	0·9761	0·2316	0·3775	0·3642	2·5202
6·8	0·8737	0·9776	0·2327	0·3885	0·3706	2·5772
7·0	0·9013	0·9789	0·2338	0·3993	0·3766	2·6326
7·2	0·9283	0·9801	0·2347	0·4100	0·3821	2·6867
7·4	0·9545	0·9812	0·2356	0·4205	0·3873	2·7394
7·6	0·9801	0·9822	0·2364	0·4308	0·3921	2·7908
7·8	1·0051	0·9831	0·2371	0·4411	0·3966	2·8410
8·0	1·0296	0·9840	0·2378	0·4511	0·4009	2·8899
8·2	1·0534	0·9848	0·2384	0·4610	0·4048	2·9378
8·4	1·0768	0·9855	0·2389	0·4708	0·4086	2·9846
8·6	1·0996	0·9862	0·2395	0·4804	0·4121	3·0303
8·8	1·1219	0·9868	0·2400	0·4899	0·4154	3·0751
9·0	1·1438	0·9874	0·2404	0·4992	0·4185	3·1189
9·2	1·1652	0·9880	0·2408	0·5084	0·4215	3·1617
9·4	1·1862	0·9885	0·2412	0·5174	0·4243	3·2037
9·6	1·2067	0·9890	0·2416	0·5263	0·4269	3·2449
9·8	1·2269	0·9894	0·2420	0·5351	0·4294	3·2853
10·0	1·2467	0·9898	0·2423	0·5438	0·4318	3·3248

TABLE 1. Leading terms of the inner solution.

Substitution of this into equations (16) enables the equations to be formally integrated one by one if taken in the appropriate order. The boundary conditions (14) are applied. For H_0 and H_1, for which no boundary conditions for $\zeta/\chi^{\frac{1}{2}} \to \infty$ are specified, the constants of integration are evaluated by comparison with the numerical solution at $9 \leqslant \zeta \leqslant 10$. The resulting asymptotic solutions, for $\frac{1}{4}\zeta^2 \gg 1$, are

$$\left.\begin{aligned}
F_1 &= -\log\zeta + \frac{1}{\alpha} - \frac{1}{\zeta^2} - \frac{5}{2\zeta^4} + \dots,\\
F_2 &= \frac{1}{2\zeta^2} + \frac{5}{2\zeta^4} + \dots,\\
F_3 &= -\frac{\log\zeta}{\zeta^2} - \frac{13\log\zeta}{2\zeta^4} + \frac{0\cdot8005}{\zeta^4} + \frac{1}{\alpha}\left(\frac{1}{\zeta^2} + \frac{5}{\zeta^4}\right) + \dots, \quad \dots;\\
G_0 &= 1 - \frac{1}{\zeta^2} - \frac{3}{2\zeta^4} + \dots,\\
G_1 &= \frac{1}{4} + \frac{1}{4\zeta^2} + \frac{9}{8\zeta^4} + \dots,\\
G_2 &= -\tfrac{1}{2}\log\zeta + \frac{1}{4} + \frac{1}{2\alpha} - \frac{1\cdot883}{\zeta^2} - \frac{0\cdot141}{\zeta^4}\\
&\qquad - \frac{3\log\zeta}{2\zeta^2} - \frac{17\log\zeta}{4\zeta^4} + \frac{1}{\alpha}\left(\frac{1}{2\zeta^2} + \frac{9}{4\zeta^4}\right) + \dots, \quad \dots;\\
H_0 &= \frac{1}{2} - \frac{2\log\zeta}{\zeta^2} - \frac{2\cdot266}{\zeta^2} + \frac{5}{\zeta^4} + \dots,\\
H_1 &= \frac{\log\zeta}{\zeta^2} + \frac{0\cdot632}{\zeta^2} - \frac{5}{\zeta^4} + \dots,\\
H_2 &= -\frac{1}{4} - \frac{\log^2\zeta}{\zeta^2} + \frac{2\log\zeta}{\zeta^2} + \frac{1\cdot133 + H_{20}}{\zeta^2} + \frac{13\log\zeta}{\zeta^4}\\
&\qquad + \frac{0\cdot851}{\zeta^4} + \frac{1}{\alpha}\left(\frac{2\log\zeta}{\zeta^2} - \frac{10}{\zeta^4}\right) + \dots, \quad \dots;\\
C_1 &= -2\log\zeta + 1 + \frac{2}{\alpha} - \frac{2}{\zeta^2} - \frac{1}{\zeta^4} + \dots,\\
C_2 &= -\log\zeta + \frac{1}{2} + \frac{1}{\alpha} + \frac{1}{2\zeta^4} + \dots,\\
C_3 &= \log^2\zeta - \left(1 + \frac{2}{\alpha}\right)\log\zeta - \frac{2\log\zeta}{\zeta^2} - \frac{5\cdot266}{\zeta^2}\\
&\qquad - \frac{2\log\zeta}{\zeta^4} + \frac{0\cdot867}{\zeta^4} + \frac{1}{\alpha\zeta^4} + \dots, \quad \dots.
\end{aligned}\right\} \tag{20}$$

Note that the constant of integration H_{20} appears in the solution for H_2 because, as is shown by (17), it cannot be evaluated by a comparison with the numerical solution until the parameters G_{20}, F_{10} and C_{10} have been determined.

By application of the asymptotic solutions (20) instead of the boundary conditions (14) all the parameters $F_{10}, F_{20}, \ldots$, can now be determined. For example instead of (19) we now have, when ζ is large,

$$F_{10}(\alpha) = F_{11} - \log \zeta + \frac{1}{\alpha} - \frac{1}{\zeta^2} - \frac{5}{2\zeta^4} + \ldots,$$

and it is found that for $9 \leqslant \zeta \leqslant 10$ the right-hand side is constant, to an accuracy consistent with the accuracy of the computed values of F_{11}. By combining the expressions (17) and (20) and making use of the numerical solution in the range $9 \leqslant \zeta \leqslant 10$ it is found that

$$\left.\begin{aligned} &F_{10} = -1{\cdot}0662 + 1/\alpha, \quad F_{20} = 0{\cdot}2500, \quad F_{30} = -0{\cdot}1444 + 1/2\alpha, \\ &G_{10} = 0{\cdot}5000, \quad G_{20} = -1{\cdot}1828 + 1/\alpha, \\ &C_{10} = -0{\cdot}3004 + 2/\alpha, \quad C_{20} = 0{\cdot}3498 + 1/\alpha, \quad C_{30} = -3{\cdot}0025 - 0{\cdot}3004/\alpha \\ &\text{and also} \quad H_{20} = -2{\cdot}99. \end{aligned}\right\} \quad (21)$$

It may be observed from the asymptotic solutions (20) that the viscous inner solution approaches the inviscid outer solution in an algebraic (or nearly algebraic) rather than exponential manner.

5. Discussion

The essential steps have been: (i) to carry out the first limiting process (7), which is an approximation of the boundary-layer type; (ii) to carry out the second limiting process (10), for which variables and an asymptotic expansion are specially chosen, and which leads to a set of ordinary differential equations; and (iii) to solve the equations, applying the outer boundary conditions through asymptotic solutions for large ζ. To complete this account the validity of the solution will be discussed and then, as a postscript, some remarks on the relation between the present solution and Hall's earlier solution will be added.

It was pointed out in § 3 that although the ordinary differential equations (16) were derived on the assumption that $\zeta = O(1)$ it might nevertheless be expected that the computed solution should be valid beyond the domain $\zeta = O(1)$. In § 4 a numerical solution and its asymptotic form were obtained, so it is now possible to consider whether the solution is, in fact, uniformly valid.

Consider the expansion (13) for F, G, H and C and its asymptotic form for large ζ given by the asymptotic solutions (20). For $\zeta = O(1)$ the expansion satisfies the equations of motion (8), or (15), and satisfies the boundary conditions on the axis. For larger ζ in the viscous core, that is, for $\zeta = O(\chi^{\frac{1}{2}})$, the expansion is still asymptotic in form, and it still satisfies the equations of motion (8); it also satisfies the requisite boundary conditions at the edge of the viscous core, as $\zeta/\chi^{\frac{1}{2}} \to \infty$. Hence the expansion (13) is valid throughout the viscous core.

Alternatively, the suitability of the expansion (13) for describing the flow throughout the viscous core may be seen as follows. The equations (8), which govern the vortex core as a whole, can be solved in two ways. First one can develop an expansion in ascending powers of ν of which the expressions (3) represent the leading term. This expansion contains terms which have algebraic or logarithmic singularities at $r \to 0$ and is therefore valid in some range of values of r which excludes the origin. Note that the upper limit of the range of validity of this expansion is not determined by the slenderness condition $(r/x)^2 \ll 1$ mentioned in § 2: this condition is necessary to reduce equations (1) to (8) but is irrelevant to the subsequent argument. Secondly, one can adopt the variables ζ and χ defined in equations (9) and expand in descending powers of χ to obtain (13). When $\zeta \gg 1$ some of the terms in this expansion are exponentially small and the others are either algebraic or logarithmic. The range of validity, in r, of this expansion is bounded above by a function of χ. Now let us replace the variables r, x in the first expansion by ζ, χ and expand it in descending powers of χ assuming

$$1 \ll \zeta \ll \chi^{\frac{1}{2}} e^{\chi}. \tag{22}$$

We find that for $\zeta \gg 1$ the new third expansion differs from the second only by terms that are exponentially small. Since the third expansion is convergent if the relation (22) holds we conclude that the expansion (13) is valid throughout the viscous core and that the first and second expansions together give a description of the entire vortex core provided only that $\chi \gg 1$. Note that while the first expansion, which is valid to the edge $\zeta = \chi^{\frac{1}{2}} e^{\chi}$ of the core, can be transformed, for $1 \ll \zeta \ll \chi^{\frac{1}{2}} e^{\chi}$, to what is essentially the second expansion, a converse also holds. The second expansion (13) can be transformed so that for $\chi^{\frac{1}{2}} \ll \zeta \leqslant \chi^{\frac{1}{2}} e^{\chi}$ it converges and can be identified with the first expansion. In a sense, therefore, the solution (13) suffices to describe the entire vortex core.

It is emphasized that important viscous effects are operative outside the domain $\zeta = O(1)$. One obvious effect is the reversal of the trend of the circumferential velocity as the axis is approached (cf. the inviscid solution and figure 1). It can be shown from the solution for G that when χ is very large $\partial G/\partial \zeta = 0$ at $\zeta \doteqdot 2\chi^{\frac{1}{2}}$: the reversal takes place well outside $\zeta = O(1)$.

The present solution appears consistent with that given earlier by Hall (1961). Hall assumed that in the viscous domain

$$u = \text{const.} + O(\epsilon u),$$

where ϵ is small. This Oseen assumption is consistent with the expansion (13) for F. Further, Hall's expression for ϵ yields $\epsilon = O(\chi^{-1} \log \zeta)$ in the present notation. Now his join with the outer solution is made at some ζ at the edge of the viscous domain and where the Oseen assumption is still valid, for which the conditions are $\zeta^2 \gg \chi$ and $\epsilon \ll 1$ respectively. Thus although one would naturally have to make the join at finite ζ for $\nu \neq 0$ one could, as ν decreases, make the join at larger and larger ζ, and in the limit $\nu \to 0$ one could make this ζ infinite. The earlier solution remains approximate in the limit $\nu \to 0$, but only because of an approximation in the radial velocity. It can also be shown that for large ζ the earlier solution for the circumferential velocity is equivalent to that obtained by putting the asymptotic solutions (20) for G_0, G_1 and G_2 in the expression (13) for G here.

There appears to be little difference in the results when the two solutions are applied to a practical problem where $\nu > 0$. In the figure, profiles of axial and circumferential velocity obtained from the present solution are superimposed on the earlier corresponding ones. The present solution is, however, easier to apply, and the attainment of greater accuracy (by the calculation of extra terms) is more straightforward.

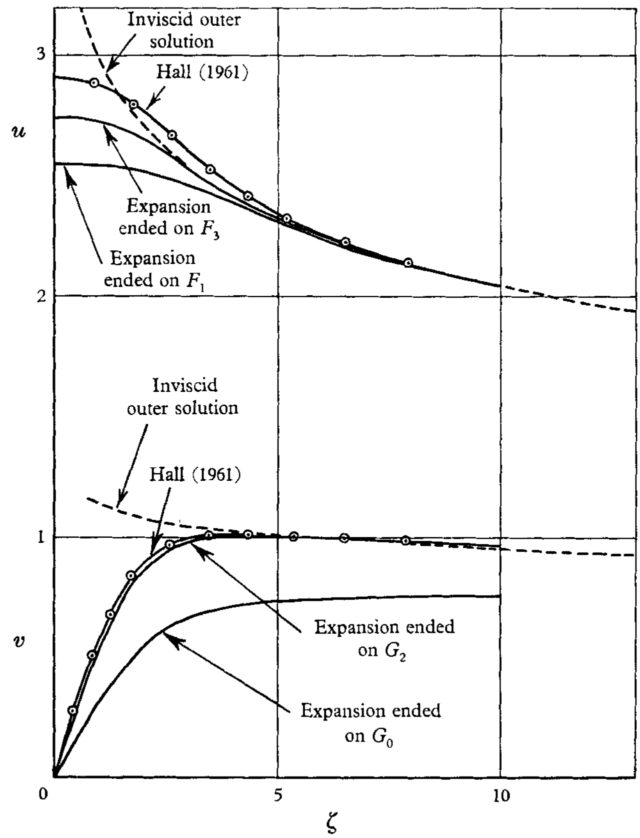

FIGURE 1. Profiles of axial (u) and circumferential (v) velocity for a vortex core at a finite Reynolds number.

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